Linear Algebra II

10/07/2020, Friday, 15:00 – 18:30 (deadline for handing in: 18.30)

- This Take-Home Exam is 'open-book', which means that the book as well as lecture notes may be used as a reference.
- For handing in the exam, the use of electronic devices is of course allowed. The student is fully responsible for handing in his/her complete work before the deadline. You are asked to upload your answers as a SINGLE **pdf-file**.
- Every student must upload the signed declaration before the start of the exam. An exam will not be graded in case the signed declaration has not been uploaded. After grading, short discussions with (a selection of) students will be held to check for possible fraud.
- The exam consists of SIX problems.
- Write your name and student number on each page!

1 $(4 + 5 + 6 = 15$ pts) **Inner product spaces**

Let P_3 be the vector space of all real polynomials $p(x)$ with degree less than or equal to 2, i.e.

$$P_3 = \{p(x) \mid p(x) = p_0 + p_1 x + p_2 x^2,\ p_i \in \mathbb{R}\}$$

For any $p(x) \in P_3$ define $\tilde{p} \in \mathbb{R}^3$ by

$$\tilde{p} := \begin{pmatrix} p(0) \\ p(1) \\ p(-1) \end{pmatrix}.$$

It was shown in the book that $\langle p(x), q(x)\rangle := \tilde{p}^T \tilde{q}$ defines an inner product on P_3.

(a) Show that the polynomials 1 and x are orthogonal.

A polynomial $p(x)$ is called *even* if $p(x) = p(-x)$ for all x, and *odd* if $p(-x) = -p(x)$ for all x. Let $\mathcal{E} \subset P_3$ be the subspace of all even polynomials, and $\mathcal{O} \subset P_3$ the subspace of all odd polynomials.

(b) Show that $\mathcal{E}$ and $\mathcal{O}$ are orthogonal.

(c) Determine an orthonormal basis of $\mathcal{E}$.

2 $(7 + 8 = 15 \text{ pts})$ **Eigenvalues**

Let A be a real $n \times n$ matrix.

(a) Suppose that λ is an eigenvalue of A with eigenvector v. Let p be a real polynomial. Show that v is also an eigenvector of $p(A)$. Determine the corresponding eigenvalue.

(b) Let p be any real polynomial such that $p(A) = 0$. Prove that every eigenvalue of A is a root of p.

3 $(3 + 3 + 3 + 2 + 4 = 15 \text{ pts})$ **Diagonalization**

In this problem, let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix.

(a) Let $M \in \mathbb{R}^{n\times n}$ be nonsingular. Show that if $A > 0$ then $M^T AM > 0$.

(b) Show that if $A > 0$ then A is nonsingular.

Now partition A as $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}$ with $A_{11} \in \mathbb{R}^{k\times k}$ and $A_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$

(c) Show that if $A > 0$ then $A_{22} > 0$.

(d) Compute the product

$$\begin{pmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ -A_{22}^{-1}A_{12}^T & I \end{pmatrix}. \tag{1}$$

(e) Show that $A > 0$ if and only if $A_{22} > 0$ and $A_{11} - A_{12}A_{22}^{-1}A_{12}^T > 0$.

4 $(3+4+4+4 = 15 \text{ pts}))$ **Unitary matrices and diagonalization**

A square complex matrix is called normal if $A^H A = AA^H$.

(a) Let A be a square complex matrix. Assume that there exists a unitary matrix U such that $U^H AU$ is a diagonal matrix. Prove that A is normal.

(b) Let T be a square complex, upper triangular matrix. Prove that T is normal if and only if T is a diagonal matrix.

(c) Let U be a unitary matrix and T an upper triangular matrix such that $U^H AU = T$. Prove that if A is normal, then also T is normal.

(d) Prove that if A is normal, then there exists a unitary matrix U such that $U^H AU$ is a diagonal matrix.

5 $(4+8+3 = 15 \text{ pts})$ **Singular value decomposition**

Let

$$M = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 1 & -2 & 3 \\ 1 & 2 & -3 \end{pmatrix}$$

(a) Determine the singular values of M

(b) Find a singular value decomposition of M.

(c) Find the best rank 2 approximation of M.

6 $(5+5+5=15$ pts) **Jordan Form**

Let $A \in \mathbb{C}^{4\times 4}$.

(a) Assume A has two distinct eigenvalues, λ_1 and λ_2, with geometric multiplicities $g_1 = 1$, $g_2 = 2$, respectively. Assume the characteristic polynomial is $p_A(z) = (z-\lambda_1)(z-\lambda_2)^3$. Determine the Jordan Form. Motivate your answer

(b) Determine the minimal polynomial of the matrix A specified in part (a). Motivate your answer.

(c) Assume now A has three distinct eigenvalues, λ_1, λ_2 and λ_3. Assume its minimal polynomial is $(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)$. Determine all possible Jordan Forms of A. Motivate your answer.

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10 pts free